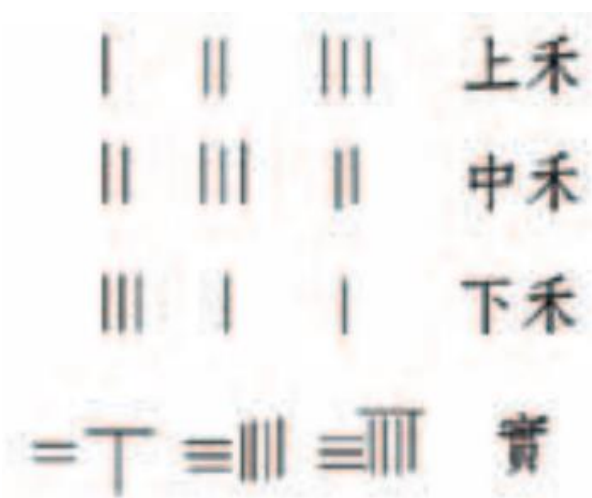
Solving System of Linear Equations

Hung-yi Lee

九章算術



九章算術卷八

晉 劉徽 注

唐 李淳風 注釋

方程以御錯糅正負

今有上禾三秉中禾二秉下禾一秉實三十九斗上禾二秉中禾三秉下禾一秉實三十四斗上禾一秉中禾二秉下禾三秉實二十六斗問上中下禾實一秉各幾何答曰上禾一秉九斗四分斗之一中禾一秉四斗四分斗之一下禾一秉二斗四分斗之三案三原本訛作一今改正

Equivalent

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solution set: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

$$\Longleftrightarrow$$

equivalent

$$\begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

Solution set: $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$

Equivalent

- Applying the following three operations on a system of linear equations will produce an equivalent one.
- 1. Interchange

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$

- 2. Scaling

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \;\; \text{X(-3)} \end{cases} \Rightarrow \begin{cases} 3x_1 & +x_2 & = 10 \\ -3x_1 & +9x_2 & = 0 \end{cases}$$

- 3. Row Addition

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \;\; \text{X(-3)} \end{cases} \Rightarrow \begin{cases} & 10x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solving system of linear equation

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.
- Strategy:

$$\begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & 10x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & x_2 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

We know how to transform the given system of linear equations into another equivalent system of linear equations.

We do it again and again until the system of linear equation is so simple that we know its answer at a glance.

Augmented Matrix

- a system of linear equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

coefficient matrix

Augmented Matrix

- a system of linear equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\quad \Rightarrow \quad A\mathbf{x} = \mathbf{b}
$$

m x (n+1)

m x n   m x 1

$$
\left[\; A \;\middle|\; \mathbf{b} \;\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}
$$

augmented matrix

Solving system of linear equation

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.
- Strategy of solving:

$$\begin{cases} x_1 - 3x_2 = 0 \\ 3x_1 + x_2 = 10 \end{cases} \quad \begin{cases} x_1 - 3x_2 = 0 \\ 10x_2 = 10 \end{cases} \quad \begin{cases} x_1 - 3x_2 = 0 \\ x_2 = 1 \end{cases} \quad \begin{cases} x_1 = 3 \\ x_2 = 1 \end{cases}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 10 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 0 \\ 0 & 10 & 10 \end{bmatrix} \quad \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

Solving system of linear equation

A ***complex*** system of linear equations

A ***simple*** system of linear equations

$$A\mathbf{x} = \mathbf{b} \quad \xleftrightarrow{\text{equivalent}} \quad A'\mathbf{x} = \mathbf{b}'$$

$$A' = [\, A \;\; \mathbf{b} \,] \rightarrow A'' \rightarrow A''' \rightarrow \cdots\cdots \rightarrow R = [\, R' \;\; \mathbf{b}' \,]$$

reduced row echelon form

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

elementary row operations

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Row Echelon Form***

1. Each nonzero row lies above every zero row

2. The leading entries are in echelon form

$$\begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Row Echelon Form***

1. Each nonzero row lies above every zero row

2. The leading entries are in echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

No zero rows

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Reduced Row Echelon Form***

1-2 The matrix is in row echelon form

3. The columns containing the leading entries are standard vectors.

$$\begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Reduced Row Echelon Form***

1-2 The matrix is in row echelon form

3. The columns containing the leading entries are standard vectors.

$$\begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

A

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix}$$

R

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading Entry

The pivot positions of A are (1,1), (2,3) and (3,4).

The pivot columns of A are 1st, 3rd and 4th columns.

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***

Example 1. Unique Solution

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{array}$$

If RREF looks like $[\ I\ \ \mathbf{b}'\ ]$

$$\begin{aligned} x_1 \qquad\qquad &= -4 \\ x_2 \qquad &= -5 \\ x_3 &= \ \ 3 \end{aligned}$$

unique solution

Example 2. Infinite Solution

$$
\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & b \\
\end{array}
$$

$$
\left[\begin{array}{cccccc}
1 & -3 & 0 & 2 & 0 & 7 \\
0 & 0 & 1 & 6 & 0 & 9 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\Rightarrow
\begin{array}{rl}
x_1 - 3x_2 \quad + 2x_4 \quad & = 7 \\
x_3 + 6x_4 \quad & = 9 \\
x_5 & = 2 \\
\cancel{0} & \cancel{= 0}
\end{array}
$$

Free variables

Basic variables

With free variables, there are infinitely many solutions.

Parametric Representation:

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 + 3x_2 - 2x_4 \\ \\ 9 - 6x_4 \\ \\ 2 \end{bmatrix}
$$

Reduced Row Echelon Form

- ***Example 3. No Solution***

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow \begin{aligned} x_1 \qquad\quad -3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

inconsistent

When an augmented matrix contains a row in which **the only nonzero entry lies in the last column**

→ The corresponding system of linear equations has **no solution** (**inconsistent**).

Gaussian Elimination

http://www.ams.org/notices/201106/rtx110600782p.pdf

- Gaussian elimination: an algorithm for finding the **reduced** **row echelon form** of a matrix.

forward pass

backward pass

Original augmented matrix

A row echelon form

The reduced row echelon form

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Elementary row operations

Elementary row operations

Please refer to the steps of Gaussian Elimination in the textbook by yourself.

http://www.dougbabcock.com/matrix.php

Example 1

$$
\begin{aligned}
x_1 \; + 2x_2 - \; x_3 + 2x_4 + \; x_5 &= 2 \\
-x_1 - 2x_2 + \; x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 \qquad\quad &= 3 \\
-3x_1 - 6x_2 + 2x_3 \qquad\qquad\quad + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c}
\\ 1 \\ -2 \\ 3
\end{array}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
-1 & -2 & 1 & 2 & 3 & 6 \\
2 & 4 & -3 & 2 & 0 & 3 \\
-3 & -6 & 2 & 0 & 3 & 9
\end{bmatrix}
$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & -1 & 6 & 6 & 15
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & -1 & 6 & 6 & 15
\end{bmatrix}
$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c c}
 & \left[\begin{array}{cccccc}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & -1 & 6 & 6 & 15
\end{array}\right] \\
\end{array}
$$

-1

Example 1

$$\begin{aligned}
x_1 \; + 2x_2 - \; x_3 + 2x_4 + \; x_5 &= 2 \\
-x_1 - 2x_2 + \; x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 \qquad\;\; &= 3 \\
-3x_1 - 6x_2 + 2x_3 \qquad\qquad + 3x_5 &= 9
\end{aligned}$$

$$\begin{array}{cc}
 & \\
 & \\
-2 & \\
 &
\end{array}
\left[\begin{array}{cccccc}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 8 & 8 & 16
\end{array}\right]$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Example 1

$$
\begin{array}{rrrrrl}
x_1 & +2x_2 & - x_3 & +2x_4 & + x_5 & =2 \\
-x_1 & -2x_2 & + x_3 & +2x_4 & +3x_5 & =6 \\
2x_1 & +4x_2 & -3x_3 & +2x_4 & & =3 \\
-3x_1 & -6x_2 & +2x_3 & & +3x_5 & =9
\end{array}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 0 & -1 & -2 \\
0 & 0 & -1 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Example 1

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array}$$
$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 \quad + \quad - x_5 &= -5 \\ x_3 \quad &= -3 \\ x_4 + x_5 &= 2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 2x_2 + x_5 \\ x_2 \\ -3 \\ 2 - x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 2x_2 + x_5 \\ x_2 \\ -3 \\ 2 - x_5 \\ x_5 \end{bmatrix}\begin{array}{c} \color{red}{-8} \\ \color{red}{1} \\ \\ \color{red}{3} \\ \color{red}{-1} \end{array} = \overset{\color{red}{1}}{x_2}\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \overset{\color{red}{-1}}{x_5}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 \quad + \quad - x_5 &= -5 \\ x_3 \qquad\qquad &= -3 \\ x_4 + x_5 &= 2 \end{aligned}$$

x_1 *free*

$$x_2 = -\frac{5}{2} - \frac{1}{2}x_1 + \frac{1}{2}x_5$$

$\cancel{x_1 = -5 - 2x_2 + x_5}$

$\cancel{x_2 \quad \text{free}}$

$x_3 = -3$

$x_4 = 2 - x_5$

$x_5 \quad \text{free}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -5/2 - 1/2x_1 + 1/2x_5 \\ -3 \\ 2 - x_5 \\ x_5 \end{bmatrix}\begin{array}{c} \color{red}{-8} \\ \color{red}{1} \\ \\ \color{red}{3} \\ \color{red}{-1} \end{array} = \overset{\color{red}{-8}}{x_1}\begin{bmatrix} 1 \\ -1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \overset{\color{red}{-1}}{x_5}\begin{bmatrix} 0 \\ 1/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -5/2 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

Example 2

- Find the RREF of

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times(-2) \quad \begin{pmatrix} R \\ \\ 2R \end{pmatrix} \Rightarrow \begin{pmatrix} R \\ \\ \boldsymbol{0} \end{pmatrix}$$

Example 3

- Find the RREF of $\begin{bmatrix} R & 2R \\ R & -R \end{bmatrix}$ $\qquad R = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} R & 2R \\ R & -R \end{bmatrix} \rightarrow \begin{bmatrix} R & 2R \\ \boldsymbol{O} & -3R \end{bmatrix} \rightarrow \begin{bmatrix} R & \boldsymbol{O} \\ \boldsymbol{O} & -3R \end{bmatrix} \rightarrow \begin{bmatrix} R & \boldsymbol{O} \\ \boldsymbol{O} & R \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF is unique

- A matrix can be transformed into multiple REF by row operation, but only one RREF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linear independent or not?

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars x_1, x_2, …, x_n, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

$Ax = \mathbf{0}$ have non-zero solution

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

Checking Independence

$$\mathcal{S} = \underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}}_{\mathbf{A}}$$

Linear independent or not?

$Ax = \mathbf{0}$ have non-zero solution or not

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{array} \right] \end{array} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \end{array}$$

Checking Independence

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \\ \\ x_4 = 0 \end{array} \longrightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = x_3 \\ x_3\ is\ free \\ x_4 = 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} - x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

setting $x_3 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

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- 感謝 同學發現投影片 上的錯誤 (zero rows)